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The diameter of a zero divisor graph

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Abstract

Let R be a commutative ring and let $Z(R)^*$ be its set of nonzero zero divisors. The set $Z(R)^*$ makes up the vertices of the corresponding zero divisor graph, $\Gamma(R)$, with two distinct vertices forming an edge if the product of the two elements is zero. The distance between vertices a and b (not necessarily distinct from a) is the length of the shortest path connecting them, and the diameter of the graph, $\mathrm{diam}(\Gamma(R))$, is the sup of these distances. For a reduced ring R with nonzero zero divisors, $1 \leqslant \mathrm{diam}(\Gamma(R)) \leqslant \mathrm{diam}(\Gamma(R[x])) \leqslant \mathrm{diam}(\Gamma(R[[x]])) \leqslant 3$. A complete characterization for the possible diameters is given exclusively in terms of the ideals of R. A similar characterization is given for $\mathrm{diam}(\Gamma(R))$ and $\mathrm{diam}(\Gamma(R[x]))$ when R is nonreduced. Various examples are provided to illustrate the difficulty in dealing with the power series ring over a nonreduced ring.

1. Introduction

Throughout this paper we assume that R denotes a commutative ring with identity and nonzero zero divisors. We let $Z(R)$ denote the set of zero divisors of R and let $Z(R)^*$ denote the (nonempty) set of nonzero zero divisors. We consider the graph $\Gamma(R)$ whose vertices are the elements of $Z(R)^*$ and whose edges are those pairs of distinct nonzero zero divisors $\{a, b\}$ such that $ab = 0$. Recall that a graph is said to be connected if for each pair of distinct vertices v and w, there is a finite sequence of distinct vertices $v_1 =$

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$v, v_2, \ldots, v_n = w$ such that each pair $\{v_i, v_{i+1}\}$ is an edge. Such a sequence is said to be a path and the distance, $d(v, w)$, between connected vertices v and w is the length of the shortest path connecting them. The diameter of a connected graph is the supremum of the distances between vertices. The diameter is 0 if the graph consists of a single vertex and a connected graph with more than one vertex has diameter 1 if and only if it is complete; i.e., each pair of distinct vertices forms an edge. In [1], D.F. Anderson and P.S. Livingston studied the graph $\Gamma(R)$. Among other things, they proved that $\Gamma(R)$ is always connected and its diameter, $\mathrm{diam}(\Gamma(R))$, is always less than or equal to 3 [1, Theorem 2.3]. They also proved that $\Gamma(R)$ is a complete graph if and only if either R is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ or $xy = 0$ for all $x, y \in Z(R)$ [1, Theorem 2.8]. More recently, M. Axtell, J. Coykendall and J. Stickles have investigated the corresponding graphs of the polynomial ring $R[\mathrm{x}]$ and the power series ring $R[\![\mathrm{x}]\!]$. For Noetherian rings, they proved that if R is not isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, then knowing that any one of $\mathrm{diam}(\Gamma(R))$, $\mathrm{diam}(\Gamma(R[\mathrm{x}]))$ or $\mathrm{diam}(\Gamma(R[\![\mathrm{x}]\!]))$ is 2 is enough to say all three graphs have diameter 2 [2, Theorem 6]. They also proved that if R is not isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, then having any one of $\Gamma(R)$, $\Gamma(R[\mathrm{x}])$ or $\Gamma(R[\![\mathrm{x}]\!])$ complete is enough to imply all three are complete [2, Theorem 3].

Our main goal in this paper is to characterize the diameter of $\Gamma(R)$, $\Gamma(R[\mathrm{x}])$ and $\Gamma(R[\![\mathrm{x}]\!])$ strictly in terms of properties of the ring R. For reduced rings, we give complete characterizations for all three graphs (see Theorem 4.9). For nonreduced rings we have succeeded only in characterizing the diameters of $\Gamma(R)$ and $\Gamma(R[\mathrm{x}])$. One of the difficulties in dealing with $R[\![\mathrm{x}]\!]$ when R is not reduced is that the zero divisors of $R[\![\mathrm{x}]\!]$ can be rather strange. For example, there is an example in [5] of a nonreduced ring R with a zero divisor of the form $r + \mathrm{x}$ in $R[\![\mathrm{x}]\!]$ (see, [5, Example 6]). Axtell, Coykendall and Stickles cite this ring as one for which $\mathrm{diam}(\Gamma(R)) = \mathrm{diam}(\Gamma(R[\mathrm{x}])) = 2$ while $\mathrm{diam}(\Gamma(R[\![\mathrm{x}]\!])) = 3$ [2, Example 1]. They leave open the existence of a reduced ring with the same sequence of diameters. In Example 5.3, we construct a reduced ring R for which $\mathrm{diam}(\Gamma(R)) = \mathrm{diam}(\Gamma(R[\mathrm{x}])) = 2$ and $\mathrm{diam}(\Gamma(R[\![\mathrm{x}]\!])) = 3$. We also give examples of both reduced (Example 5.1) and nonreduced (Example 5.2) rings R where $\mathrm{diam}(\Gamma(R)) = 2$ and $\mathrm{diam}(\Gamma(R[\mathrm{x}])) = \mathrm{diam}(\Gamma(R[\![\mathrm{x}]\!])) = 3$. In the latter, the ring R is constructed using the idealization of a module. Recall that for a ring T and T-module B, the idealization of B is the ring $T(+)B$ built from the product $T \times B$ by setting $(r, b) + (s, c) = (r + s, b + c)$ and $(r, b)(s, c) = (rs, rc + sb)$ (see, for example, the chapter on examples in [7]).

For a polynomial or power series $f = \sum f_i \mathrm{x}^i$, we use $c(f)$ to denote the ideal of R generated by the coefficients of f.

2. General characterizations

We start by establishing necessary and sufficient conditions for $\mathrm{diam}(\Gamma(R))$ to be a particular number. Anderson and Livingston have characterized when $\mathrm{diam}(\Gamma(R)) = 0$ and when $\mathrm{diam}(\Gamma(R)) = 1$ [1, Theorem 2.8]. Axtell, Coykendall and Stickles have several results involving necessary conditions for the diameter of $\Gamma(R)$ to be (less than or) equal to 2.

In general, the behavior of the zero divisors in nonreduced rings is not quite the same as it is in reduced ones. The main differences stem from the fact that a nonzero zero divisor in a reduced ring must be contained in at least one minimal prime, but cannot be contained in each minimal prime. In particular, this implies that if R is reduced, then $Z(R)$ is the union of the minimal primes. Thus it helps to first establish some basic properties for zero divisors in reduced rings and later establish similar properties for nonreduced rings. Note that for reduced rings, those with exactly two minimal primes constitute a special case that we must deal with. For nonreduced rings, those where $Z(R)^2 = (0)$ form a somewhat special case. It turns out that away from these special cases, the characterizations for the values of $\operatorname{diam}(\Gamma(R))$ and $\operatorname{diam}(\Gamma(R[\mathrm{x}]))$ are the same no matter whether R is reduced or nonreduced. Theorems 2.1 and 2.4 are quite similar, but the proofs used are specific to whether the ring in question is reduced (Theorem 2.1) or not (Theorem 2.4).

We first consider reduced rings. Recall that if I is a finitely generated ideal of a reduced ring R, then I has a nonzero annihilator if and only if I is contained in at least one minimal prime. The basic scheme for the proof is as follows. First, if $r \neq 0$ annihilates I, then since R is reduced, r cannot be contained in each minimal prime. But with $rI = (0)$, each minimal prime that does not contain r must contain I. Conversely, if some minimal prime P contains I, then $IR_P = PR_P = (0)$ since R_P is a 0-dimensional reduced ring; i.e., it is a field. Since I is finitely generated, there is an element $t \in R\backslash P$ such that $tI = (0)$. For an ideal J that is not finitely generated, we have that J has a nonzero annihilator if and only if the intersection of the minimal primes that do not contain J is nonzero. Note that the latter condition implies that J is contained in some minimal prime, but this alone is not enough to guarantee that J has a nonzero annihilator.

In our first result we provide a sufficient condition for $\Gamma(R)$ to have diameter 3 when R is a reduced ring. Later we show that this condition is also necessary. A similar equivalence holds for nonreduced rings, but in this case the number of minimal primes is irrelevant.

Theorem 2.1. *Let R be a reduced ring. If R has more than two minimal primes and there are nonzero elements $a, b \in Z(R)$ such that (a, b) has no nonzero annihilator, then* $\operatorname{diam}(\Gamma(R)) = 3$.

Proof. If R contains a pair of zero divisors a and b, such that $ab \neq 0$ and $(0 : (a, b)) = (0)$, then $d(a, b) = 3$ and therefore $\operatorname{diam}(\Gamma(R)) = 3$. Thus we may assume R has more than two minimal primes and that there are nonzero elements $a, b \in Z(R)$ such that $ab = 0$ and $(0 : (a, b)) = (0)$. Each minimal prime of R contains at least one of a and b, but with $(0 : (a, b)) = (0)$, none contains both. Thus without loss of generality we may assume there are minimal primes P, Q and N such that $a \in (Q \cap N)\backslash P$ and $b \in P\backslash(Q \cup N)$. Let $q \in (Q \cap P)\backslash N$ and consider the pair $a + bq$ and b. Since R is reduced, $ab = 0$ and neither b nor q is contained in N, $0 \neq bq^2 = b(a + bq)$. On the other hand, $(a, b) = (a + bq, b)$ is an ideal with no nonzero annihilator. Thus $d(a + bq, b) = 3$ and $\operatorname{diam}(\Gamma(R)) = 3$. □

The next result follows easily from Theorem 2.1. It and statement (3) of Theorem 2.6 are closely related to several results in [2]. One difference here is our restriction to reduced rings, another is their restriction to those rings R where it is first assumed that $\operatorname{diam}(\Gamma(R)) = 2$.

Theorem 2.2. *Let R be a reduced ring. If $Z(R)$ is not an ideal, then the diameter of $\Gamma(R)$ is less than or equal to 2 if and only if R has exactly two minimal primes.*

Proof. Assume $Z(R)$ is not an ideal. Then there are elements $a, b \in Z(R)$ such that $a+b$ is not a zero divisor, and therefore the ideal (a, b) has no nonzero annihilators. Also, since R is reduced it must have at least two minimal primes, say P and Q. Moreover, $Z(R)$ is the union of the minimal primes of R.

If R has more than two minimal primes, then $\operatorname{diam}(\Gamma(R)) = 3$ by Theorem 2.1. Conversely, if P and Q are the only minimal primes of R, then $Z(R) = P \cup Q$ and we may assume $a \in P \backslash Q$ and $b \in Q \backslash P$. Obviously, $ab \in P \cap Q = (0)$ since R is reduced. Let $r, s \in Z(R)$ be distinct elements. Since no nonzero element can be in both P and Q, either $rs = 0$ or exactly one of P and Q contains the ideal (r, s). If $rs = 0$, then $d(r, s) = 1$. On the other hand, if $rs \neq 0$, then $br = 0 = bs$ if $P \supseteq (r, s)$ and $ar = 0 = as$ if $Q \supseteq (r, s)$. Hence $d(r, s) = 2$ if $rs \neq 0$. It follows that $\operatorname{diam}(\Gamma(R)) \leqslant 2$. $\square$

If R is a reduced ring with exactly two minimal primes, then $Z(R)$ cannot be an ideal since it is the union of the minimal primes. Thus the initial assumption that $Z(R)$ is not an ideal is not needed to show $\operatorname{diam}(\Gamma(R)) = 2$ when R has exactly two minimal primes. On the other hand, the reduced ring R in Example 5.1 has infinitely many minimal primes but $\operatorname{diam}(\Gamma(R)) = 2$, so necessarily, $Z(R)$ is an ideal (which must be prime). Thus the assumption that $Z(R)$ is not an ideal does play a significant role in establishing the converse in Theorem 2.2. On the other hand, it is not entirely necessary. A slight modification in the construction used in Example 5.1 will yield a reduced ring whose set of zero divisors forms an ideal, yet the diameter of the zero divisor graph is three (see the comment after the proof of Example 5.1).

The following elementary result has likely been noticed before.

Lemma 2.3. *Let R be a nonreduced ring and let I be an ideal of R. If I has a nonzero annihilator and $q \in \operatorname{Nil}(R)$, then the ideal $qR + I$ has a nonzero annihilator. In particular, if $a \in Z(R)$ and $q \in \operatorname{Nil}(R)$, then $a + q \in Z(R)$ and $(0 : (a, q)) \neq (0)$.*

Proof. Let q be a nonzero nilpotent and assume $cI = (0)$ where $c \neq 0$. Since q is nilpotent, there is a positive integer m such that $cq^m = 0$ with $cq^{m-1} \neq 0$. Clearly, cq^{m-1} is a nonzero annihilator of $qR + I$. $\square$

The next result is the nonreduced version of Theorem 2.1 above. Note that here we do not assume that R has more than two minimal primes.

Theorem 2.4. *Let R be a nonreduced ring. If there is a pair of zero divisors $a, b \in Z(R)$ such that $(0 : (a, b)) = (0)$, then $\operatorname{diam}(\Gamma(R)) = 3$.*

Proof. Let $a, b \in Z(R)$ be such that $(0 : (a, b)) = (0)$. Then $d(a, b) \neq 2$. By the previous lemma, neither a nor b can be nilpotent. If $ab \neq 0$, then $d(a, b) = 3$. Thus we may assume $ab = 0$. Since $ab = 0$, $(a, b)^2 = (a^2, b^2)$ has no nonzero annihilator. Thus without loss of generality we may assume there is a nilpotent q such that $b^2 q \neq 0$. Since a is a zero divisor and q is nilpotent, $a+bq$ is a zero divisor by Lemma 2.3. Consider the pair $a+bq$ and b. Since $(a,b)=(a+bq,b)$ has no nonzero annihilator, $d(a+bq,b)\neq 2$. But $(a+bq)b=b^2q\neq 0$. Thus $d(a+bq,b)=3$ and $\operatorname{diam}(\Gamma(R))=3$. □

Corollary 2.5. *If R is a nonreduced ring such that $Z(R)$ is not an ideal, then* $\operatorname{diam}(\Gamma(R))=3$.

Our next result characterizes the diameter of $\Gamma(R)$ in terms of the ideals of R. The first statement is directly from [1] and the second is, more or less, simply the definition of having $\operatorname{diam}(\Gamma(R))=1$.

Theorem 2.6. *Let R be a ring.*

(1) $\operatorname{diam}(\Gamma(R))=0$ *if and only if R is* (*nonreduced and*) *isomorphic to either $\mathbb{Z}_4$ or* $\mathbb{Z}_2[\mathrm{y}]/(\mathrm{y}^2)$.
(2) $\operatorname{diam}(\Gamma(R))=1$ *if and only if $xy=0$ for each distinct pair of zero divisors and R has at least two nonzero zero divisors.*
(3) $\operatorname{diam}(\Gamma(R))=2$ *if and only if either* (i) *R is reduced with exactly two minimal primes and at least three nonzero zero divisors, or* (ii) *$Z(R)$ is an ideal whose square is not (0) and each pair of distinct zero divisors has a nonzero annihilator.*
(4) $\operatorname{diam}(\Gamma(R))=3$ *if and only if there are zero divisors $a\neq b$ such that* $(0:(a,b))=(0)$ *and either* (i) *R is a reduced ring with more than two minimal primes, or* (ii) *R is nonreduced.*

Proof. The statement in (1) can be found in [1].

With regard to (2), Anderson and Livingston proved that $\operatorname{diam}(\Gamma(R))=1$ if and only if either (i) R is (reduced and) isomorphic to $\mathbb{Z}_2\times\mathbb{Z}_2$, or (ii) R is nonreduced, $Z(R)^2=(0)$ and R is not isomorphic to either $\mathbb{Z}_4$ or $\mathbb{Z}_2[\mathrm{y}]/(\mathrm{y}^2)$ [1, Theorem 2.8].

For (3), we start with reduced rings. First, if R is reduced and has exactly two nonzero zero divisors, then each must annihilate the other, so $\operatorname{diam}(\Gamma(R))=1$. By Theorem 2.8 of [1], this occurs only when R is isomorphic to $\mathbb{Z}_2\times\mathbb{Z}_2$. Combining this with Theorem 2.2, shows that if R is reduced with exactly two minimal primes and more than two zero divisors, then $\operatorname{diam}(\Gamma(R))=2$.

If there are (nonzero) zero divisors $a,b\in Z(R)$ such that $(0:(a,b))=(0)$ and either R is nonreduced or is reduced with more that two minimal primes, then $\operatorname{diam}(\Gamma(R))=3$ by Theorems 2.1 and 2.4.

On the other hand, if each pair of zero divisors has a nonzero annihilator, then $Z(R)$ is an ideal and $\operatorname{diam}(\Gamma(R))\leqslant 2$. Combining this with the characterizations of when $\operatorname{diam}(\Gamma(R))\leqslant 1$ finishes the proof of statement (3).

Finally statement (4) is from Theorems 2.1 and 2.4 and (1)–(3). □

While these characterizations are relatively simple observations, they are quite useful when considering polynomial rings and power series rings.

3. Polynomial rings

For polynomials, McCoy's Theorem states that a polynomial $f(\mathrm{x}) \in R[\mathrm{x}]$ is a zero divisor if and only if there is a nonzero element $r \in R$ such that $rf(\mathrm{x}) = 0$. Based on this theorem, a ring R is said to be a *McCoy ring* if each finitely generated ideal contained in $Z(R)$ has a nonzero annihilator [4] (referred to as Property A in [7], [8] and [9]).

We first recall a result due to Y. Quentel for reduced rings [10, Proposition 6] and to J. Huckaba and J. Keller for nonreduced rings [8, Theorem 1].

Theorem 3.1. *The polynomial ring* $R[\mathrm{x}]$ *is a McCoy ring.*

As a corollary we have the following useful result. We have stated it only for the nontrivial case that neither polynomial is 0.

Corollary 3.2. *If* $f(\mathrm{x})$ *and* $g(\mathrm{x})$ *are nonzero zero divisors of* $R[\mathrm{x}]$*, then the following are equivalent.*

(1) $(f(\mathrm{x}), g(\mathrm{x})) \subseteq Z(R[\mathrm{x}])$.
(2) $f(\mathrm{x})$ *and* $g(\mathrm{x})$ *have a common nonzero annihilator in* $R[\mathrm{x}]$.
(3) *There is a nonzero element* $r \in R$ *such that* $rf(\mathrm{x}) = 0 = rg(\mathrm{x})$.
(4) *If* $deg(f(\mathrm{x})) = n$, *then* $f(\mathrm{x}) + \mathrm{x}^{n+1}g(\mathrm{x})$ *is a zero divisor of* $R[\mathrm{x}]$.

Theorem 3.3. *For a ring* R, $Z(R[\mathrm{x}])$ *is an ideal of* $R[\mathrm{x}]$ *if and only if* R *is a McCoy ring such that* $Z(R)$ *is an ideal.*

Proof. A consequence of McCoy's Theorem is that it is always the case that $Z(R[\mathrm{x}])$ is contained in $Z(R)[\mathrm{x}]$. Also $Z(R[\mathrm{x}])$ always contains $Z(R)$. Thus if $Z(R[\mathrm{x}])$ is an ideal, then $Z(R[\mathrm{x}]) = Z(R)[\mathrm{x}]$. But this means that for any finite set of zero divisors in R, any polynomial whose coefficients are contained in this set must be a zero divisor. Thus each such set must have a nonzero annihilator. Hence $Z(R)$ must be an ideal and R must be a McCoy ring.

Conversely, suppose that R is a McCoy ring and $Z(R)$ is an ideal. Then each finite subset of $Z(R)$ has a nonzero annihilator and each polynomial whose coefficients are contained in $Z(R)$ must be a zero divisor of $R[\mathrm{x}]$. It follows that if $f(\mathrm{x})$ and $g(\mathrm{x})$ are a pair of nonzero zero divisors where $deg(f(\mathrm{x})) = n$, then $f(\mathrm{x}) + \mathrm{x}^{n+1}g(\mathrm{x})$ is a zero divisor of $R[\mathrm{x}]$. Hence by Corollary 3.2, $(f(\mathrm{x}), g(\mathrm{x})) \subseteq Z(R[\mathrm{x}])$ and we have that $Z(R[\mathrm{x}])$ is an ideal. $\square$

With this we have enough to characterize the diameter of $\Gamma(R[\mathrm{x}])$ based on the ideals of R.

Theorem 3.4. *Let* R *be a ring.*

(1) $\mathrm{diam}(\Gamma(R[\mathrm{x}]) \geqslant 1$.
(2) $\mathrm{diam}(\Gamma(R[\mathrm{x}])) = 1$ *if and only if* R *is a nonreduced ring such that* $Z(R)^2 = (0)$.

(3) $\operatorname{diam}(\Gamma(R[\mathrm{x}])) = 2$ *if and only if either* (i) *R is a reduced ring with exactly two minimal primes, or* (ii) *R is a McCoy ring and $Z(R)$ is an ideal with $Z(R)^2 \neq (0)$.*
(4) $\operatorname{diam}(\Gamma(R[\mathrm{x}])) = 3$ *if and only if R is not a reduced ring with exactly two minimal primes and either R is not a McCoy ring or $Z(R)$ is not an ideal.*

Proof. The first statement holds because the only rings with $\operatorname{diam}(\Gamma(T)) = 0$ are those rings T that are isomorphic to either $\mathbb{Z}_4$ or $\mathbb{Z}_2[\mathrm{y}]/(\mathrm{y}^2)$ [1]. The second is a combination of [1, Theorem 2.8] and [2, Theorem 3].

For (3), we can quickly dispatch with the case that R is reduced with exactly two minimal primes. Denote these primes by P and Q. Then $R[\mathrm{x}]$ has exactly two minimal primes, namely $P[\mathrm{x}]$ and $Q[\mathrm{x}]$. Clearly, $R[\mathrm{x}]$ is not isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ (or simply note that both $P[\mathrm{x}]$ and $Q[\mathrm{x}]$ contain infinitely many members). Thus $\operatorname{diam}(\Gamma(R[\mathrm{x}])) = 2$ by Theorem 2.6.

For reduced rings with more than two minimal primes and nonreduced rings, we know that $\operatorname{diam}(\Gamma(T)) = 2$ if and only if $Z(T)^2 \neq (0)$ and each pair of zero divisors has a nonzero annihilator (Theorem 2.6). It follows that if R is either a reduced ring with more than two minimal primes or a nonreduced ring, then $\operatorname{diam}(\Gamma(R[\mathrm{x}])) = 2$ if and only if $Z(R[\mathrm{x}])$ is an ideal whose square is not the zero ideal and each pair of zero divisors of $R[\mathrm{x}]$ has a nonzero annihilator. But by Corollary 3.2, $Z(R[\mathrm{x}])$ is an ideal if and only if $(f(\mathrm{x}), g(\mathrm{x}))$ has a nonzero annihilator for each pair $f(\mathrm{x}), g(\mathrm{x}) \in R[\mathrm{x}]$. Thus $\operatorname{diam}(\Gamma(R[\mathrm{x}])) = 2$ if and only if $Z(R[\mathrm{x}])$ is an ideal of $R[\mathrm{x}]$ whose square is not the zero ideal. By Theorem 3.3, the latter occurs if and only if R is a McCoy ring with $Z(R)$ an ideal (of R) such that $Z(R)^2 \neq (0)$.

For the final statement, note that if $Z(R)^2 = (0)$, then $Z(R)$ is an ideal and R is a nonreduced McCoy ring. Also $Z(R)$ is not an ideal if R is reduced with exactly two minimal primes (but R is a McCoy ring in this case). The result follows. □

If the total quotient ring of R is von Neumann regular, then the only primes that contain only zero divisors are the minimal primes. Hence $Z(R)$ is an ideal only in the trivial case that R is an integral domain, a case we have assumed does not happen.

Corollary 3.5. *Let R be a reduced ring that is not isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. If the total quotient ring of R is von Neumann regular, then* $\operatorname{diam}(\Gamma(R)) = \operatorname{diam}(\Gamma(R[\mathrm{x}]))$. *Moreover,* $\operatorname{diam}(\Gamma(R)) = 3$ *if and only if R has more than two minimal primes.*

A slightly different interpretation of Theorem 3.4 seems in order. Namely, for which rings can we have $\operatorname{diam}(\Gamma(R)) < \operatorname{diam}(\Gamma(R[\mathrm{x}]))$. As noted in [2], it is clear that $\operatorname{diam}(\Gamma(R))$ is always less than or equal to $\operatorname{diam}(\Gamma(R[\mathrm{x}]))$. For convenience, we also include all of the cases where $\operatorname{diam}(\Gamma(R)) = \operatorname{diam}(\Gamma(R[\mathrm{x}]))$.

Theorem 3.6. *Let R be a ring. The following cases describe all possibilities for the pair* $\operatorname{diam}(\Gamma(R))$, $\operatorname{diam}(\Gamma(R[\mathrm{x}]))$.

(1) $\operatorname{diam}(\Gamma(R)) = 0$ *and* $\operatorname{diam}(\Gamma(R[\mathrm{x}])) = 1$ *if and only if R is isomorphic to either $\mathbb{Z}_4$ or $\mathbb{Z}_2[\mathrm{y}]/(\mathrm{y}^2)$.*

(2) $\operatorname{diam}(\Gamma(R)) = \operatorname{diam}(\Gamma(R[\mathrm{x}])) = 1$ *if and only if R is a nonreduced ring with more than one nonzero zero divisor such that* $Z(R)^2 = (0)$.
(3) $\operatorname{diam}(\Gamma(R)) = 1$ *and* $\operatorname{diam}(\Gamma(R[\mathrm{x}])) = 2$ *if and only if R is isomorphic to* $\mathbb{Z}_2 \times \mathbb{Z}_2$.
(4) $\operatorname{diam}(\Gamma(R)) = \operatorname{diam}(\Gamma(R[\mathrm{x}])) = 2$ *if and only if either* (i) *R is a reduced ring with exactly two minimal primes and R has more than two nonzero zero divisors, or* (ii) $Z(R)$ *is an ideal with* $Z(R)^2 \neq (0)$ *and R is a McCoy ring.*
(5) $\operatorname{diam}(\Gamma(R)) = 2$ *and* $\operatorname{diam}(\Gamma(R[\mathrm{x}])) = 3$ *if and only if $Z(R)$ is an ideal, R is not a McCoy ring but each pair of zero divisors of R has a nonzero annihilator.*
(6) $\operatorname{diam}(\Gamma(R)) = \operatorname{diam}(\Gamma(R[\mathrm{x}])) = 3$ *if and only if R is not a reduced ring with exactly two minimal primes and there is a pair of zero divisors a and b such that* $(0 : (a, b)) = (0)$.

Proof. The statement in (1) is due to the combination of the Anderson/Livingston characterizations of when $\operatorname{diam}(\Gamma(R)) \leqslant 1$ and Theorem 3 of [2]. Another application of the same results together with Proposition 3 of [2] takes care of statements (2) and (3).

For statements (4) and (5), we know that if R is a reduced ring with exactly two minimal primes, then $\operatorname{diam}(\Gamma(R[\mathrm{x}])) = 2$ (Theorem 2.6). The other situation where we know $\operatorname{diam}(\Gamma(R[\mathrm{x}])) = 2$ is when R is a McCoy ring such that $Z(R)$ is an ideal whose square is not the zero ideal (Theorem 3.4). To also have $\operatorname{diam}(\Gamma(R)) = 2$, either (i) R is a reduced ring with exactly two minimal primes and at least three nonzero zero divisors, or (ii) $Z(R)$ is an ideal with $Z(R)^2 \neq (0)$ and each pair of zero divisors has a nonzero annihilator (Theorem 2.6). If R is McCoy and $Z(R)$ is an ideal, then each finite subset of $Z(R)$ has a nonzero annihilator. Thus $\operatorname{diam}(\Gamma(R)) = \operatorname{diam}(\Gamma(R[\mathrm{x}])) = 2$ if and only if either (i) R is a reduced ring with exactly two minimal primes and more than two nonzero zero divisors, or (ii) $Z(R)$ is an ideal with $Z(R)^2 \neq (0)$ and R is a McCoy ring. As noted earlier, if $Z(R)^2 = (0)$ or R is reduced with exactly two minimal primes, then R is a McCoy ring. Thus $\operatorname{diam}(\Gamma(R)) = 2$ and $\operatorname{diam}(\Gamma(R[\mathrm{x}])) = 3$ if and only if R is not a McCoy ring and $Z(R)$ is an ideal such that each pair of zero divisors has a nonzero annihilator.

Finally both diameters are three if and only if R is not a reduced ring with exactly two minimal primes and there is a pair of zero divisors $a, b \in Z(R)$ such that $(0 : (a, b)) = (0)$ (no matter whether $Z(R)$ is an ideal or not). $\square$

4. Power series rings

For power series, we have been able to completely characterize the diameter of $\Gamma(R[\![\mathrm{x}]\!])$ in terms of the ideals of R when R is reduced. For nonreduced rings we only know a few special cases.

In [5], D.E. Fields presented an example, due to R. Gilmer, of a nonreduced ring S with a (nonzero) zero divisor s such that $s + \mathrm{x}$ is a zero divisor of $S[\![\mathrm{x}]\!]$ [5, Example 3]. On the other hand, a result of Gilmer, A. Grams and T. Parker shows that a power series $r(\mathrm{x})$ over a reduced ring R is a zero divisor if and only if there is a nonzero element $t \in R$ such that $tr(\mathrm{x}) = 0$ [6, Proposition 3.5]. For a power series ring $R[\![\mathrm{x}]\!]$ in a single indeterminate, their result says that if $f = \sum f_i \mathrm{x}^i$ and $g = \sum g_i \mathrm{x}^i$ are power series for which $fg = 0$, then the product $c(f)c(g)$ is contained in the nilradical of R. Thus, if R is reduced, we have $fg = 0$

if and only if $f_i g_j = 0$ for each pair of coefficients f_i and g_j. As we will see, this result is key in determining the diameter of $\Gamma(R[\![x]\!])$ when R is a reduced ring. The first theorem of this section simply puts the results of [6] in a convenient form.

Theorem 4.1. *Let R be a reduced ring. If $r(x) = \sum r_i x^i$ is a zero divisor, then there is a nonzero $s \in R$ such that $sr_i = 0$ for each i. Moreover, for $s(x) \in R[\![x]\!]$, $r(x)s(x) = 0$ if and only if $r_i s_j = 0$ for each i and j.*

Corollary 4.2. *Let R be a reduced ring and let $I = (f(x), g(x))$ be an ideal of the power series ring $R[\![x]\!]$. Then the following are equivalent.*

(1) *The ideal $I = (f(x), g(x))$ has a nonzero annihilator in $R[\![x]\!]$.*
(2) *There is a nonzero element $a \in R$ such that $aI = (0)$.*
(3) *The ideal $c(f) + c(g)$ has a nonzero annihilator in R.*

Proof. Clearly, if $c(f) + c(g)$ has a nonzero annihilator in R, then both $f(x)$ and $g(x)$ are zero divisors of $R[\![x]\!]$ with a common nonzero annihilator in R. So (3) implies both (1) and (2). It is also clear that (2) implies (1). To finish the proof, assume $r(x) = \sum r_i x^i$ is a nonzero annihilator of I. Then by [6, Proposition 3.5] (Theorem 4.1), each r_i must annihilate each coefficient of both $f(x)$ and $g(x)$. It follows that $c(f) + c(g)$ has nonzero annihilator in R. $\square$

Here is the power series version of statement (3) of Theorem 3.4 for reduced rings (see also [2, Theorems 4, 5 and 6]).

Theorem 4.3. *Let R be a reduced ring with exactly two minimal primes, P and Q. Then* $\operatorname{diam}(\Gamma(R[\![x]\!])) = 2$.

Proof. Since R is reduced, $Z(R) = P \cup Q$ and $Z(R[\![x]\!]) \subset Z(R)[\![x]\!]$. Moreover, $P \cap Q = PQ = (0)$. Thus a power series $a(x) = \sum a_i x^i$ is a zero divisor if and only if $(a_0, a_1, \ldots)$ is contained in at least one of P and Q—and only one if $a(x) \neq 0$. It follows that $Z(R[\![x]\!]) = P[\![x]\!] \cup Q[\![x]\!]$ with $P[\![x]\!]Q[\![x]\!] = (0)$. Hence $\operatorname{diam}(\Gamma(R[\![x]\!])) = 2$. $\square$

Since $\operatorname{diam}(\Gamma(R[\![x]\!])) \geqslant \operatorname{diam}(\Gamma(R))$, if the diameter of $\Gamma(R)$ is 3, then the same is true for $\Gamma(R[\![x]\!])$. This observation together with Theorem 2.2 gives us the following.

Theorem 4.4. *Let R be a reduced ring with more than two minimal primes. If $Z(R)$ is not an ideal, then* $\operatorname{diam}(\Gamma(R[\![x]\!])) = 3$.

The ring R in Example 5.4 is a reduced ring such that $Z(R)$ is an (nonzero prime) ideal and $\operatorname{diam}(\Gamma(R[\![x]\!])) = 2$.

We know that if there is a two generated ideal contained in $Z(R)$ with no nonzero annihilator, then $\operatorname{diam}(\Gamma(R)) = 3$. Also we know that if $Z(R)$ is an ideal and R is not a McCoy ring, then there are polynomials $f(x), g(x) \in Z(R[x]) = Z(R)[x]$ with

$d(f(\mathrm{x}), g(\mathrm{x})) = 3$. Viewed in $R[\![\mathrm{x}]\!]$ we will still have $d(f(\mathrm{x}), g(\mathrm{x})) = 3$ when R is reduced since $f(\mathrm{x})$ and $g(\mathrm{x})$ have no common annihilator in R. Our next result characterizes when $\mathrm{diam}(R[\![\mathrm{x}]\!]) = 2$ for a reduced ring such that $Z(R)$ is an ideal.

Theorem 4.5. *Let R be a reduced ring such that $Z(R)$ is an ideal of R. Then the diameter of $\Gamma(R[\![\mathrm{x}]\!])$ is 2 if and only if $I + J$ has a nonzero annihilator whenever I and J are both countably generated with nonzero annihilators.*

Proof. Since R is reduced, each zero divisor of $R[\![\mathrm{x}]\!]$ has a nonzero annihilator in R. Thus $Z(R[\![\mathrm{x}]\!]) \subset Z(R)[\![\mathrm{x}]\!]$.

Assume $\mathrm{diam}(\Gamma(R[\![\mathrm{x}]\!]) = 2$. Since $Z(R)$ is an ideal and R is reduced, R has infinitely many minimal primes. So by either Theorem 2.2 or Theorem 2.6, $Z(R[\![\mathrm{x}]\!])$ is an ideal of $R[\![\mathrm{x}]\!]$. Let $I = (a_0, a_1, \ldots)$ and $J = (b_0, b_1, \ldots)$ be countably generated ideals of R with nonzero annihilators. Consider the power series $a(\mathrm{x}) = \sum a_i \mathrm{x}^{2i}$ and $b(\mathrm{x}) = \sum b_i \mathrm{x}^{2i+1}$. Both $a(\mathrm{x})$ and $b(\mathrm{x})$ are zero divisors of $R[\![\mathrm{x}]\!]$. Thus so is the sum $a(\mathrm{x}) + b(\mathrm{x})$. Clearly the coefficients of $a(\mathrm{x}) + b(\mathrm{x})$ generate the ideal $I + J$. Hence $I + J$ must have a nonzero annihilator in R.

For the converse, assume that whenever I and J are countably generated ideals with nonzero annihilators, then $I + J$ has a nonzero annihilator.

Let $a(\mathrm{x}) = \sum a_i \mathrm{x}^i$ and $b(\mathrm{x}) = \sum b_i \mathrm{x}^i$ be nonzero zero divisors in $R[\![\mathrm{x}]\!]$. Then the ideals $I = (a_0, a_1, \ldots)$ and $J = (b_0, b_1, \ldots)$ are countably generated ideals with nonzero annihilators. Thus $I + J$ has a nonzero annihilator. Clearly the ideal $(a_0 + b_0, a_1 + b_1, \ldots)$ is contained in $I + J$. Hence it has a nonzero annihilator. It follows that $a(\mathrm{x}) + b(\mathrm{x})$ has a nonzero annihilator as does the ideal $(a(\mathrm{x}), b(\mathrm{x}))$. Thus $Z(R[\![\mathrm{x}]\!])$ is an ideal of $R[\![\mathrm{x}]\!]$ where each two generated subideal has a nonzero annihilator. Therefore $\mathrm{diam}(\Gamma(R[\![\mathrm{x}]\!])) = 2$ by Theorem 2.6. □

Note that if $I + J$ has a nonzero annihilator whenever I and J are countably generated ideals with nonzero annihilators, then R will be a McCoy ring and $Z(R)$ will be an ideal.

Corollary 4.6. *Let R be a reduced ring. If $\mathrm{diam}(\Gamma(R[\![\mathrm{x}]\!])) = 2$, then R is a McCoy ring and $\mathrm{diam}(\Gamma(R[\mathrm{x}])) = 2$.*

Proof. Assume $\mathrm{diam}(R[\![\mathrm{x}]\!]) = 2$. If $Z(R)$ is not an ideal, then $R[\![\mathrm{x}]\!]$ must have exactly two minimal primes by Theorem 2.2. As each minimal prime of $R[\![\mathrm{x}]\!]$ is the extension of a minimal prime of R, R has exactly two minimal primes. In this case, R is a McCoy ring and $\mathrm{diam}(R[\mathrm{x}]) = 2$ by Theorem 3.4. On the other hand, if $Z(R)$ is an ideal and I is finitely generated, then I is a finite sum of principal ideals, each with a nonzero annihilator. Thus, by Theorem 4.5, I must have a nonzero annihilator. So in this case R is a McCoy ring and $\mathrm{diam}(R[\mathrm{x}]) = 2$ (Theorem 3.6). □

A ring R is said to be *countably McCoy* if each ideal $I \subset Z(R)$ that is countably generated has a nonzero annihilator.

Corollary 4.7. *Let R be a reduced ring. If $Z(R)$ is an ideal of R and R is countably McCoy, then* $\operatorname{diam}(\Gamma(R[\![x]\!])) = 2$.

Corollary 4.8. *Let R be a reduced ring. Then* $\operatorname{diam}(\Gamma(R[\![x]\!])) = 2$ *if and only if either R has exactly two minimal primes or R is a McCoy ring where $Z(R)$ is an ideal and $I + J$ has a nonzero annihilator whenever I and J are countably generated ideals of R with nonzero annihilators.*

We will present our partial results concerning nonreduced power series in our last section which is primarily devoted to examples. We end this section by collecting all of the various results concerning diameters of the graphs $\Gamma(R)$, $\Gamma(R[x])$ and $\Gamma(R[\![x]\!])$ for R a reduced ring.

Theorem 4.9. *Let R be a reduced ring that is not an integral domain. Then* $1 \leqslant \operatorname{diam}(\Gamma(R)) \leqslant \operatorname{diam}(\Gamma(R[x])) \leqslant \operatorname{diam}(\Gamma(R[\![x]\!])) \leqslant 3$. *Moreover, here are all possible sequences for these dimensions.*

(1) $\operatorname{diam}(\Gamma(R)) = 1$ *and* $\operatorname{diam}(\Gamma(R[x])) = \operatorname{diam}(\Gamma(R[\![x]\!])) = 2$ *if and only if R is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.*
(2) $\operatorname{diam}(\Gamma(R)) = \operatorname{diam}(\Gamma(R[x])) = \operatorname{diam}(\Gamma(R[\![x]\!])) = 2$ *if and only if either R has exactly two minimal primes and is not isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ or for each pair of countably generated ideals I and J with nonzero annihilators, the sum $I + J$ has a nonzero annihilator* (*and R is a McCoy ring with $Z(R)$ an ideal*).
(3) $\operatorname{diam}(\Gamma(R)) = \operatorname{diam}(\Gamma(R[x])) = 2$ *and* $\operatorname{diam}(\Gamma(R[\![x]\!])) = 3$ *if and only if R is a McCoy ring with $Z(R)$ an ideal but there exists countably generated ideals I and J with nonzero annihilators such that $I + J$ does not have a nonzero annihilator.*
(4) $\operatorname{diam}(\Gamma(R)) = 2$ *and* $\operatorname{diam}(\Gamma(R[x])) = \operatorname{diam}(\Gamma(R[\![x]\!])) = 3$ *if and only if $Z(R)$ is an ideal and each two generated ideal contained in $Z(R)$ has a nonzero annihilator but R is not a McCoy ring.*
(5) $\operatorname{diam}(\Gamma(R)) = \operatorname{diam}(\Gamma(R[x])) = \operatorname{diam}(\Gamma(R[\![x]\!])) = 3$ *if and only if R has more than two minimal primes and there is a pair of zero divisors a and b such that (a, b) does not have a nonzero annihilator.*

5. Examples

We will start by exhibiting both a reduced ring and a (similar) nonreduced ring where the diameter of the zero divisor graph of the ring is two while the diameter of that for the corresponding polynomial ring is three.

Both of these examples are based on the domain $D = K[w, y, z]_M$ where K is a field, w, y and z are algebraically independent indeterminates and $M = (w, y, z)K[w, y, z]$. The constructions are similar to ones employed in [9] to construct rings where each two generated ideal containing only zero divisors has a nonzero annihilator, but some three generated ideal (containing only zero divisors) does not. For the nonreduced ring, R is built using idealization. For the reduced ring we use a variation on the so-called "$A + B$" construction (see, for example, [7] and/or the previously cited [9]).

We start with the reduced ring.

Example 5.1. Let Q be the maximal ideal of $D = K[\mathrm{w}, \mathrm{y}, \mathrm{z}]_{(\mathrm{w},\mathrm{y},\mathrm{z})}$ and let $\mathcal{P}$ denote the set of height two primes of D. For each $P_\alpha \in \mathcal{P}$, let $Q_\alpha = Q/P_\alpha$. Let $\mathcal{I} = \mathcal{A} \times \mathbb{N}$ where $\mathcal{A}$ is an index set for $\mathcal{P}$ and let $B = \sum Q_i$ where $Q_i = Q_\alpha$ for each $i = (\alpha, n) \in \mathcal{I}$. Finally let $R = D + B$ be the ring constructed from $D \times B$ by setting $(r, a) + (s, b) = (r + s, a + b)$ and $(r, a)(s, b) = (rs, rb + sa + ab)$.

(1) $Z(R) = Q + B$ is a prime ideal of R.
(2) Each two generated ideal contained in $Z(R)$ has a nonzero annihilator.
(3) R is not a McCoy ring.
(4) $\operatorname{diam}(\Gamma(R)) = 2$ and $\operatorname{diam}(\Gamma(R[\mathrm{x}])) = \operatorname{diam}(\Gamma(R[\![\mathrm{x}]\!])) = 3$.

Proof. For each $i = (\alpha, n) \in \mathcal{I}$, each $r \in D$ and each $b \in B$, let r_i denote the image of r in D/P_α and let b_i denote the ith component of b. Since D is local, r_i is a unit (of D/P_α) if and only if r is a unit of D. Also note that b_i is in Q/P_α, the Jacobson radical of D/P_α. Hence if r is a unit of D, then $r_i + b_i$ is a unit of D/P_α.

Let $r \in Q$ and $b \in B$ with at least one not 0. Since D is a local Noetherian domain of dimension 3, r is contained in infinitely many height two primes. Thus $rQ_i = (0)$ for infinitely many Q_i. Since b has at most finitely many nonzero components, there is a Q_i such that the ith component of b is zero and $rQ_i = (0)$. Choose an element c of B whose ith component is nonzero with all other components equal to 0. Then $(r, b)(0, c) = (0, 0)$. Hence (r, b) is a zero divisor of R.

Now assume (s, f) is a zero divisor of R. There is nothing to prove if $s \in Q$, so we may assume s is a unit of D. Since s is a unit of D, $(s, f)(t, a) = (0, 0)$ implies $t = 0$. So we have $(s, f)(0, a) = (0, sa + fa) = (0, 0)$. But by the above, $s_i + f_i$ is unit for each $i = (\alpha, n)$. Thus $(sa + fa)_i = (s_i + f_i)a_i = 0$ implies $a_i = 0$ for each i. Hence (s, f) is not a zero divisor if s is a unit of D. It follows that $Z(R) = Q + B$ is an ideal of R.

Let $(r, a), (t, c)$ be a distinct pair of zero divisors of R. Then both r and t are in Q. Since D is Noetherian, there is a height two prime P_α that contains both r and t. It may be that only one such P_α contains both r and t, but there are infinitely many $i \in \mathcal{I}$ of the form $i = (\alpha, n)$. As above, both a and c have finitely many nonzero components. Thus there are infinitely many i for which $r_i = t_i = a_i = c_i = 0$. Choose one such i and take $d \in B$ with $d_i \neq 0$ and all other components equal to 0. Then $(r, a)(0, d) = (0, 0) = (t, c)(0, d)$. Therefore each two generated ideal in $Z(R)$ has a nonzero annihilator.

Consider the ideal $J = ((\mathrm{w}, 0), (\mathrm{y}, 0), (\mathrm{z}, 0)) \subset Q + B$. For each Q_α, $(\mathrm{w}, \mathrm{y}, \mathrm{z})Q_\alpha = (Q^2 + P_\alpha)/P_\alpha \neq (0)$. Hence, J has no nonzero annihilator in R. Since $Q + B = Z(R)$, R is not a McCoy ring. Therefore by Theorem 3.6 above, $\operatorname{diam}(\Gamma(R)) = 2$ and $\operatorname{diam}(\Gamma(R[\mathrm{x}])) = 3$. $\square$

A similar construction based on using the height one primes for the set $\mathcal{P}$ will yield a reduced ring R with $Z(R)$ an ideal and $\operatorname{diam}(\Gamma(R)) = 3$ instead of 2.

For the nonreduced version, we do not need infinitely many copies of each Q_α.

Example 5.2. As above, let $\mathcal{P}$ denote the height two primes of $D = K[\mathrm{w}, \mathrm{y}, \mathrm{z}]_{(\mathrm{w},\mathrm{y},\mathrm{z})}$ and let Q be the maximal ideal of D, but now let $B = \sum F_\alpha$ where $F_\alpha = qf(D/P_\alpha)$ for each $P_\alpha \in \mathcal{P}$. Let $R = D(+)B$ be the idealization of B over D.

(1) R is a local ring with maximal ideal $Q(+)B = Z(R)$.
(2) Each two generated ideal contained in $Z(R)$ has a nonzero annihilator but R is not a McCoy ring.
(3) $\mathrm{diam}(\Gamma(R)) = 2$ but $\mathrm{diam}(\Gamma(R[\mathrm{x}])) = \mathrm{diam}(\Gamma(R[\![\mathrm{x}]\!])) = 3$.

Proof. If $t \in D$ is a unit, then so is (t, b) for each $b \in B$. The inverse is the element $(t^{-1}, -t^{-2}b)$. Since D is an integral domain, if $r \neq 0$ is in D, then (r, c) is a zero divisor if and only if there is a nonzero $f \in B$ such that $rf = 0$. This occurs for each nonunit of D since each such element is contained in a height two prime. Thus each nonunit of R is a zero divisor and $Q(+)B = Z(R) = \{(r, a) \mid a \in B$ and r a nonunit of $D\}$. Since $(\mathrm{w}, 0)(\mathrm{y}, 0) \neq (0, 0)$, $\mathrm{diam}(\Gamma(R)) \geqslant 2$.

Let $(r, b), (s, c) \in Z(R)$. Since D is Noetherian, some height two prime contains both r and s. Let P_α be such a prime and let $e \in B$ be the element whose α component is 1 with all other components 0. Then $(r, b)(0, e) = (0, re) = (0, 0) = (0, se) = (s, c)(0, e)$. Thus each two generated ideal contained in $Z(R)$ has a nonzero annihilator. It follows that $\mathrm{diam}(\Gamma(R)) = 2$.

On the other hand, the ideal $((\mathrm{y}, 0), (\mathrm{w}, 0), (\mathrm{z}, 0))$ contains only zero divisors but it does not have a nonzero annihilator since no P_α contains all three of y, w and z. Thus $\mathrm{diam}(\Gamma(R)) = 2$ and $\mathrm{diam}(\Gamma(R[\mathrm{x}])) = 3$ by Theorem 3.6. □

It is possible to construct a reduced ring R that is not countably McCoy but does have $\mathrm{diam}(\Gamma(R)) = \mathrm{diam}(\Gamma(R[\![\mathrm{x}]\!])) = 2$ with $Z(R)$ an ideal. We construct such a ring later. Before doing so, we construct a reduced McCoy ring R where $Z(R)$ is an ideal, so $\mathrm{diam}(\Gamma(R)) = \mathrm{diam}(\Gamma(R[\mathrm{x}])) = 2$, but with $\mathrm{diam}(\Gamma(R[\![\mathrm{x}]\!])) = 3$.

Both rings will be constructed in a manner somewhat similar to the construction used for Example 5.1 above. For both we start with the domain $D = F[\mathcal{X}]_{(\mathcal{X})}$ where F is a field and $\mathcal{X} = \{\mathrm{x}_n\}$ is a countably infinite set of indeterminates. We let $M (= (\mathcal{X})D)$ denote the maximal ideal of D. Next we let $\mathcal{P}$ denote the primes of D that are generated by finite subsets of $\mathcal{X}$. The set $\mathcal{P}$ includes $P_0 = (0)$, the prime generated by the empty subset of $\mathcal{X}$. Also for $n \geqslant 1$, we let $P_n = (\mathrm{x}_1, \mathrm{x}_2, \ldots, \mathrm{x}_n)D$. Note that given a prime $P_\alpha \in \mathcal{P}$, there is an integer n such that $P_\alpha \subset P_k$ for each $k \geqslant n$. For each $P_\alpha \in \mathcal{P}$, we let $Q_\alpha = M/P_\alpha$.

Example 5.3. For the domain D above, let $\mathcal{O} = \{P_n \mid n \geqslant 0\}$ and let $C = \sum Q_n$. Let $R = D + C$ be the ring formed from the product $D \times C$ by setting $(r, b) + (s, c) = (r + s, b + c)$ and $(r, b)(s, c) = (rs, rc + sb + bc)$. Then R is a reduced McCoy ring such that $Z(R)$ is an ideal so $\mathrm{diam}(\Gamma(R)) = \mathrm{diam}(\Gamma(R[\mathrm{x}])) = 2$. However, $\mathrm{diam}(\Gamma(R[\![\mathrm{x}]\!])) = 3$ since $Z(R[\![\mathrm{x}]\!])$ is not an ideal.

Proof. We start by setting some notation. For each integer $n \geqslant 0$ and each $r \in D$ and $c \in C$, let $(r)_n$ denote the image of r in $D_n = D/P_n$ and let $(c)_n$ denote the component of c in Q_n. Note that each element of M corresponds to a unique element of C since $Q_0 = M$.

Also each nonzero element of M is in all but finitely many P_n's. Finally, we let $y(m)$ be the element of C for which $(y(m))_n = 0$ for $n \neq m$ and $(y(m))_m = (\mathrm{x}_{m+1})_m$ (the image of x_{m+1} in Q_m).

Consider an element of the form (r, b) where r is a unit of D. We will show that such an element is a not a zero divisor of R. Obviously, if $s \in D$ is not zero, then for each $c \in C$, $(r, b)(s, c) \neq (0, 0)$. Suppose $(r, b)(0, c) = (0, 0)$ for some $c \in C$. Then $(r)_m(c)_m + (b)_m(c)_m = 0$ for each m. Since $(b)_m \in M/P_m$ and $(r)_m$ is a unit of the local ring D_m, $(r)_m + (b)_m$ is a unit of D_m. It follows that $(c)_m = 0$ for each m and therefore (r, b) is not a zero divisor. Thus $M + C$ contains $Z(R)$.

Let $\{(f_1, b_1), (f_2, b_2), \ldots, (f_m, b_m)\}$ be a finite subset of $M + C$. For a sufficiently large integer n, each f_j is in P_k and $(b_j)_k = 0$ for all $k \geqslant n$. It is clear that $(0, y(n))$ is a nonzero annihilator of each (f_j, b_j). Hence not only is R a McCoy ring, but $Z(R) = M + C$ is an ideal of R. Since R is reduced, $\operatorname{diam}(\Gamma(R)) = \operatorname{diam}(\Gamma(R[\mathrm{x}])) = 2$ by Theorems 2.6 and 3.4.

Consider the countably generated ideal $A = ((0, y(0)), (0, y(1)), (0, y(2)), \ldots)$. Since $Q_0 = M$ and $(y(0))_0 = \mathrm{x}_1$ is not zero, no nonzero element of the form $(r, 0)$ annihilates $(0, y(0))$. However, $(-\mathrm{x}_1, y(0))$ does annihilate $(0, y(0))$ and all of the other $(0, y(i))$'s. Thus A has a nonzero annihilator. On the other hand, no nonzero element of the form $(0, s)$ annihilates A and $(r, t)(\mathrm{x}_1, 0) = (0, 0)$ implies $r = 0$. Thus the ideal $A + (\mathrm{x}_1, 0)R$ is a countably generated ideal contained in $Z(R)$ that has no nonzero annihilator, but both A and $(\mathrm{x}_1, 0)R$ have nonzero annihilators. That $\operatorname{diam}(\Gamma(R[\![\mathrm{x}]\!])) = 3$ now follows from Theorem 4.9. $\square$

The construction for our next example is much more complicated. What we will construct is a reduced McCoy ring R for which $Z(R)$ is an ideal and the sum of any two countably generated ideals with nonzero annihilators will have a nonzero annihilator but where some countably generated ideal containing only zero divisors will not have a nonzero annihilator. Thus each of the graphs $\Gamma(R)$, $\Gamma(R[\mathrm{x}])$ and $\Gamma(R[\![\mathrm{x}]\!])$ will have diameter 2 and R will have infinitely many minimal primes, but R will not be countably McCoy.

As above, we start with $D = F[\mathcal{X}]_{(\mathcal{X})}$ where $\mathcal{X} = \{\mathrm{x}_n\}$ and $\mathcal{P}$ as the set of primes generated by finite subsets of $\mathcal{X}$. Let $\mathcal{N} = \{P_\alpha\}$ denote the set of nonzero primes of $\mathcal{P}$. Let $\mathcal{A}$ be an index set for $\mathcal{N}$ and let $\mathcal{I} = \mathcal{A} \times \mathbb{R}^+$ (with $\mathbb{R}^+$ the positive reals). Let $E = \prod D_i$ where $D_i = D_\alpha = D/P_\alpha$ for each $i = (\alpha, r)$, also let $Q_i = Q_\alpha = M/P_\alpha$. We let $\mathcal{I}^o = \{i = (\alpha, r) \in \mathcal{I} \mid r \leqslant 1\}$ and for each $i = (\alpha, r) \in \mathcal{I}^o$, we let $\bar{r} = \{t \in \mathbb{R}^+ \mid t - r \in \mathbb{Z}\}$ and $\bar{i} = \{j \in \mathcal{I} \mid j = (\alpha, t)$ for some $t \in \bar{r}\}$. For each nonempty finite subset X of $\mathcal{X}$ and each positive integer n, let $\mathrm{X}_{(n)} = \{\mathrm{x}_{nk} \mid \mathrm{x}_k \in \mathrm{X}\}$. Obviously, for a pair of finite sets X and Y, $\mathrm{X} \subset \mathrm{Y}$ if and only if $\mathrm{X}_{(n)} \subset \mathrm{Y}_{(n)}$ for each (equivalently some) n. Note that $\mathrm{X}_{(1)} = \mathrm{X}$. For a nonzero polynomial $b \in F[\mathcal{X}]$ with constant term 0, there is a minimal finite subset $\mathrm{X} \subset \mathcal{X}$ such that $b(\mathrm{X}) \in F[\mathrm{X}]$. For such a $b = b(\mathrm{X})$ and minimal set X, if $\mathrm{X} \subset \mathrm{Y}$, then we also have $b(\mathrm{Y}) \in F[\mathrm{Y}]$. Moreover, we also have $b(\mathrm{X}_{(n)}) = b(\mathrm{Y}_{(n)})$ for each n using the substitution of x_{nk} for each x_k in X (and Y). For a fixed $i = (\alpha, r) \in \mathcal{I}^o$, let $b(i)$ denote the element of E where $(b(i))_j = 0$ if $j \notin \bar{i}$ and $(b(i))_j = (b(\mathrm{X}_{(n+1)}))_j$ if $j = (\alpha, r + n) \in \bar{i}$. Since the constant term of b is 0, $b(i) \in \prod Q_i$. Note that for a fixed $i \in \mathcal{I}^o$ and two such polynomials $b(\mathrm{X})$ and $d(\mathrm{X})$, $b(i)d(i) = (bd)(i)$. Let $B \subset E$ be the D-subalgebra of E consisting of the finite sums of the form $d = s_1b_1(i_1) + s_2b_2(i_2) + \cdots + s_mb_m(i_m)$ with each $s_k \in D$, each $b_k \in \mathcal{X}F[\mathcal{X}]$ and each $i_k \in \mathcal{I}^o$. Since each sum is finite, there is a finite subset X such that each s_k is in $F[\mathrm{X}]_{(\mathrm{X})}$ and each $b_k \in F[\mathrm{X}]$. Let $R = D + B$ be the ring formed from the product $D \times B$ using the same definition for addition and multiplication as that used in the previous example.

Example 5.4. Let D and $R = D + B$ be the rings defined in the previous paragraph. The ring R has the following properties.

(1) R is a reduced McCoy ring.
(2) $Z(R) = M + B$ is an ideal of R.
(3) R is not countably McCoy, but if I and J are countably generated ideals with nonzero annihilators, then $I + J$ has a nonzero annihilator.
(4) $\operatorname{diam}(\Gamma(R)) = \operatorname{diam}(\Gamma(R[\mathrm{x}])) = \operatorname{diam}(\Gamma(R[\![\mathrm{x}]\!])) = 2$.

Proof. For each $f \in D$ and each $i = (\alpha, r) \in \mathcal{I}$, we let $(f)_i$ denote the image of f in D_i and for each $b \in B$, we let $(b)_i$ denote the ith component of b.

Let $f \in M$ and let $P_\alpha \in \mathcal{P}$ be such that $f \in P_\alpha$. Then for each positive real number r, $(f)_i = 0$ for $i = (\alpha, r)$. Since each $b \in B$ has only countably many nonzero components, there is a positive real number r with $r \leqslant 1$ such that $(b)_j = 0$ for each $j \in \bar{i}$ (and $i = (\alpha, r)$). Choose one such r and corresponding $i = (\alpha, r)$, then consider $c(i)$ for $c = \mathrm{x}_1$. From the above, $c(i)$ is the element of B for which $(c(i))_j = 0$ if $j \notin \bar{i}$ and $(c(i))_j = (\mathrm{x}_n)_j$ (the image of x_n in $D_j = D_\alpha$) for $j = (\alpha, r + n - 1) \in \bar{i}$. For n sufficiently large, $(\mathrm{x}_n)_j \neq 0$ in D_α. Thus $c(i)$ is not zero. Clearly, the product $(f, b)(0, c(i))$ is $(0, 0)$, the zero element of R. Thus each element of $M + B$ is a zero divisor of R.

To see that $Z(R) = M + B$, let $t \in D$ be a unit and let $d \in B$. Then $(t)_i$ is a unit in each D_i and $(d)_i$ is in Q_i, the Jacobson radical of D_i. Hence $(t)_i + (d)_i$ is a unit of D_i. It follows that (t, d) is not a zero divisor of R. Therefore $Z(R) = M + B$ is an ideal of R.

Before going further, we need to examine the zero divisors in a little more detail. Let $b \in \mathrm{X}F[\mathrm{X}]$ be a nonzero polynomial with constant term 0 and X be the minimal subset of $\mathcal{X}$ for b. If P_α contains the set X, then the image of b in D_α is 0. On the other hand, if P_α does not contain X, then the image of b in D_α is nonzero. More generally, for $i = (\alpha, r) \in \mathcal{I}^o$ and $j = (\alpha, r + n) \in \bar{i}$ with $n \geqslant 0$, $(b(i))_j = 0$ if $\mathrm{X}_{(n+1)}$ is in P_α and $(b(i))_j \neq 0$ otherwise. Let $d \in B$ be nonzero. Then there are finitely many nonzero polynomials $d_1, d_2, \ldots, d_m \in \mathcal{X}F[\mathcal{X}]$, finitely many rational expressions $s_1, s_2, \ldots, s_m \in D$ and finitely many elements $i_1 = (\alpha_1, r_1), i_2 = (\alpha_2, r_2), \ldots, i_m = (\alpha_m, r_m) \in \mathcal{I}^o$ such that $d = s_1 d_1(i_1) + s_2 d_2(i_2) + \cdots + s_m d_m(i_m)$. We also have a minimal finite set X such that each d_k and s_k is in $F[\mathrm{X}]_{(\mathrm{X})}$. Note that for sufficiently large n, $F[\mathrm{X}]_{(\mathrm{X})} \cap F[\mathrm{X}_{(n)}]_{(\mathrm{X}_{(n)})} = F$. Because of this, the only way $(d)_j$ can be nonzero for some j is for it to be nonzero for all but finitely many members of whatever $\bar{i}$ contains j. On the other hand, $(d)_j$ is 0 for all but countably many j and there are only finitely many $i \in \mathcal{I}^o$ such that $(d)_j \neq 0$ for some $j \in \bar{i}$. In particular, for each $\alpha \in \mathcal{A}$ there are uncountably many $i = (\alpha, r) \in \mathcal{I}^o$ such that $(d)_j = 0$ for each $j \in \bar{i}$.

Choose a $d \in B$ and a nonzero $t \in M$ and consider the element (t, d). First, each element in the annihilator of (t, d) is of the form $(0, e)$ for some $e \in B$. Let $i = (\alpha, r) \in \mathcal{I}^o$. If $t \in P_\alpha$, $(t)_j = 0$ for each $j \in \bar{i}$. Thus $(te + de)_j = 0$ implies $(d)_j(e)_j = 0$. Hence for each such j, at least one of $(d)_j$ and $(e)_j$ is 0. Moreover, either $(d)_j$ or $(e)_j$ is 0 for infinitely many such j. Thus either $(d)_j = 0$ for all $j \in \bar{i}$ or $(e)_j = 0$ for all $j \in \bar{i}$. On the other hand, if t is not in P_α, then for each $j = (\alpha, s)$, $(t)_j \neq 0$. Let Y be a finite set for which $t \in \mathrm{Y}F[\mathrm{Y}]_{(\mathrm{Y})}$. For $i = (\alpha, r) \in \mathcal{I}^o$ and sufficiently large n, $\mathrm{Y} \cap \mathrm{X}_{(n+1)}$ is empty and therefore $(t)_j + (d)_j$ is a unit in $(D_\alpha)_{(\mathrm{X}_{(n+1)})}$ for $j = (\alpha, r+n) \in \bar{i}$. Since $((t)_j + (d)_j)(e)_j = (te + de)_j = 0$ for each such j, $(e)_j = 0$ for infinitely many, and hence all, $j \in \bar{i}$.

Let $I = ((s_1, d_1), (s_2, d_2), \ldots, (s_n, d_n))$ be a finitely generated ideal with each (s_m, b_m) in $Z(R)$. Since $Z(R) = M + B$ is an ideal, I is contained in $Z(R)$. Note that the ideal $(s_1, s_2, \ldots, s_n)$ is contained in infinitely many P_α's. Since there are at most finitely many $i \in \mathcal{I}^o$ for which some $(d_m)_j$ is not zero for some $j \in \bar{i}$, there are uncountably many $i \in \mathcal{I}^o$ such that $(s_m)_j = (d_m)_j = 0$ for each m and each $j \in \bar{i}$. Choose one such i and, as above, consider $c(i)$ where $c = \mathrm{x}_1$. Since $(s_m)_j = (d_m)_j = 0$ for each $j \in \bar{i}$, $(0, c(i))$ is a nonzero annihilator of the ideal I. Thus I has a nonzero annihilator and from this we have that R is a McCoy ring. Hence $\operatorname{diam}(\Gamma(R)) = \operatorname{diam}(\Gamma(R[\mathrm{x}])) = 2$ by Theorem 3.4.

Since M is countably generated and in no P_α, MR is a countably generated ideal contained in $Z(R)$ which has no nonzero annihilator. Thus R is not countably McCoy.

Let I be a countably generated ideal of R contained in $M + B$ and generated by the countable set $\{(s_m, d_m)\}$. We first consider the case that for each prime ideal P_α, there is an s_m which is not contained in P_α. By the argument above, if $(0, e)$ is an annihilator of (s_m, d_m), then for each $j = (\alpha, s) \in \mathcal{I}$, $(e)_j = 0$. Since we have assumed no P_α contains every s_m, I must have no nonzero annihilators.

Now consider what happens if some P_α does contain each s_m. Since I is countably generated and each d_m is dependent on only finitely many i's in $\mathcal{I}^o$, there are uncountably many $i = (\alpha, r) \in \mathcal{I}^o$ such that $(d_m)_j = 0$ for each $j \in \bar{i}$. As with a finitely generated ideal, choose one such i and again let $c(i)$ be "generated" by $c = \mathrm{x}_1$. The element $(0, c(i))$ is a nonzero annihilator of I.

Let I and J be a pair of countably generated ideals with nonzero annihilators. Let $\{(s_m, d_m)\}$ be a countable generating set for I and let $\{(t_m, e_m)\}$ be a countable generating set for J. Then there are primes P_α and P_β with P_α containing each s_m and P_β containing each t_m. Since each prime is generated by a finite subset of $\mathcal{X}$, the union of the these two generating sets generates a prime P_γ that contains both the s_m's and the t_m's. It follows that $I + J$ will have a nonzero annihilator. Therefore $\operatorname{diam}(\Gamma(R[\![\mathrm{x}]\!])) = 2$ by Theorem 4.5. $\square$

It is much more difficult to deal with $R[\![\mathrm{x}]\!]$ when R has nonzero nilpotents. Here are three relatively simple examples to illustrate some of the difficulties. The first is a slight variation on an example that appears in [6] (see also [5, Example 1]).

Example 5.5. Let $D = K[\mathcal{Y}, \mathcal{Z}]$ where K is a field of characteristic 2 and $\mathcal{Y} = \{\mathrm{y}_n\}_{n=0}^\infty$ and $\mathcal{Z} = \{\mathrm{z}_n\}_{n=0}^\infty$ are disjoint countably infinite sets of indeterminates. Let $R = D/A$ where $A = (\mathrm{y}_0\mathrm{z}_0, \{\mathrm{y}_0\mathrm{y}_n\}, \{\mathrm{y}_n^2\}, \{\mathrm{z}_0\mathrm{z}_n\}, \{\mathrm{z}_n^2\})D$.

(1) R is a zero-dimensional ring with a single maximal (minimal) prime, $M = (\mathcal{Y}, \mathcal{Z})R$.
(2) For each nonzero $r \in M$, $r^2 = 0$.
(3) For each pair of zero divisors r and s, either $rs = 0$ or rs is a nonzero annihilator of the ideal (r, s).

(4) R is a McCoy ring, $Z(R)$ is an ideal and $\operatorname{diam}(\Gamma(R)) = \operatorname{diam}(\Gamma(R[\mathrm{x}])) = 2$.
(5) $Z(R[\![\mathrm{x}]\!]) = M[\![\mathrm{x}]\!]$ is an ideal of $R[\![\mathrm{x}]\!]$. Moreover, $f^2 = 0$ for each $f \in M[\![\mathrm{x}]\!]$.
(6) $\operatorname{diam}(\Gamma(R[\![\mathrm{x}]\!])) = 2$ even though there are countably generated ideals I and J each with a nonzero annihilator such that $I + J$ has no nonzero annihilator.

Proof. Clearly, R is zero-dimensional and $M = (\mathcal{Y}, \mathcal{Z})R$ is its unique prime ideal. Since K has characteristic 2 and A contains the square of each indeterminate, $r^2 = 0$ for each $r \in M$. Thus for a pair of elements $r, s \in M$, either $rs = 0$ or rs is a nonzero annihilator of (r, s). For a finitely generated ideal $I = (r_1, r_2, \ldots, r_n)$ either $I^2 = 0$ or some nonzero product of its generators is a nonzero annihilator of I. Thus R is a McCoy ring and $Z(R) = M$ is an ideal of R. The graph $\Gamma(R)$ is not complete since, for example, $\mathrm{y}_1\mathrm{z}_1$ is not in A. Thus $\operatorname{diam}(\Gamma(R)) > 1$. So by Theorem 3.4, $\operatorname{diam}(\Gamma(R)) = \operatorname{diam}(\Gamma(R[\mathrm{x}])) = 2$.

For a power series $f = \sum f_i \mathrm{x}^i$, if f_k is not a zero divisor of R while $f_0, f_1, \ldots, f_{k-1}$ are all nilpotent, then f is not a zero divisor of $R[\![\mathrm{x}]\!]$ [5, Lemma 2]. Since M is both the maximal ideal and the nilradical of R we have that $Z(R[\![\mathrm{x}]\!])$ is contained in $M[\![\mathrm{x}]\!]$. Let $f = \sum f_i \mathrm{x}^i \in M[\![\mathrm{x}]\!]$. Then $f^2 = \sum f_i^2 \mathrm{x}^{2i} = 0$ since K has characteristic 2 and $r^2 = 0$ for each $r \in M$. Thus for a pair of power series $f, g \in M[\![\mathrm{x}]\!]$ we either have that $fg = 0$ or we have that fg is a nonzero annihilator of both f and g. So not only do we have $Z(R[\![\mathrm{x}]\!]) = M[\![\mathrm{x}]\!]$, we also have $\operatorname{diam}(\Gamma(R[\![\mathrm{x}]\!])) = 2$.

Denote the images of y_n and z_n in R by y_n and z_n, respectively. Consider the countably generated ideals $I = (y_0, y_1, y_2, \ldots)R$ and $J = (z_0, z_1, z_2, \ldots)R$. Each ideal has a nonzero annihilator, $y_0 I = (0)$ and $z_0 J = (0)$, but the sum $I + J = M$ does not have a nonzero annihilator. As a consequence we have that $h(\mathrm{x}) = \sum y_i \mathrm{x}^i$ and $k(\mathrm{x}) = \sum z_i \mathrm{x}^i$ are nonzero zero divisors of $R[\![\mathrm{x}]\!]$ with $h(\mathrm{x})k(\mathrm{x}) \neq 0$. Thus $h(\mathrm{x})k(\mathrm{x})$ is a common nonzero annihilator of $h(\mathrm{x})$ and $k(\mathrm{x})$, but there is no common nonzero annihilator of this pair in R. This is in contrast to what happens for reduced rings (see either Theorem 4.1 or Theorem 4.9). $\square$

Example 5.6. Let $R = \mathbb{Z}(+)\mathbb{Z}_p$, the idealization of the integers modulo some prime p, and let $S = \mathbb{Z}(+)\mathbb{Z}(p^\infty)$.

(1) $Z(R) = p\mathbb{Z}(+)\mathbb{Z}_p$ is a maximal ideal of R.
(2) $Z(S) = p\mathbb{Z}(+)\mathbb{Z}(p^\infty)$ is a maximal ideal of S.
(3) Both $Z(R)$ and $Z(S)$ have nonzero annihilators. Also $Z(R)\operatorname{Nil}(R) = (0)$, but $Z(S)\operatorname{Nil}(S) \neq (0)$.
(4) Both R and S are McCoy rings.
(5) $\operatorname{diam}(\Gamma(R)) = \operatorname{diam}(\Gamma(R[\mathrm{x}])) = \operatorname{diam}(\Gamma(R[\![\mathrm{x}]\!])) = 2$.
(6) $\operatorname{diam}(\Gamma(S)) = \operatorname{diam}(\Gamma(S[\mathrm{x}])) = 2$ but $\operatorname{diam}(\Gamma(S[\![\mathrm{x}]\!])) = 3$.

Proof. Statements (1) and (2) are clear and (4) follows from the fact that each of the maximal ideals $Z(R)$ and $Z(S)$ has nonzero annihilators, $(0, 1)$ annihilates $Z(R)$ and $(0, \overline{(1/p)})$ annihilates $Z(S)$. That $\operatorname{diam}(\Gamma(R)) = \operatorname{diam}(\Gamma(R[\mathrm{x}])) = 2$ and $\operatorname{diam}(\Gamma(S)) = \operatorname{diam}(\Gamma(S[\mathrm{x}])) = 2$ now follow easily from Theorem 3.4. Also, since R is a Noetherian ring with $\operatorname{diam}(\Gamma(R)) = 2$, $\operatorname{diam}(\Gamma(R[\![\mathrm{x}]\!])) = 2$ by [2, Theorem 4].

Since $p\mathbb{Z} \cdot \mathbb{Z}_p = (0)$, $Z(R)\operatorname{Nil}(R) = (0)$. On the other hand, $(p, 0)(0, \overline{(1/p^2)}) = (0, \overline{(1/p)}) \neq (0, 0)$. Hence $Z(S)\operatorname{Nil}(S) \neq (0)$.

Consider the polynomial $(p,0)-(1,0)\mathrm{x}$. In $R[\![\mathrm{x}]\!]$, this polynomial is not a zero divisor, but in $S[\![\mathrm{x}]\!]$ it is. In particular, it annihilates $(0,\overline{(1/p)})+(0,\overline{(1/p^2)})\mathrm{x}+(0,\overline{(1/p^3)})\mathrm{x}^2+(0,\overline{(1/p^4)})\mathrm{x}^3+\cdots$. Clearly $(1,0)\mathrm{x}$ is not a zero divisor, so $Z(S[\![\mathrm{x}]\!])$ is not an ideal. Thus $\operatorname{diam}(\Gamma(S[\![\mathrm{x}]\!]))=3$ by Corollary 2.5. □

As a $\mathbb{Z}$-module, $\mathbb{Z}(p^\infty)$ is divisible; i.e., for each nonzero $r\in\mathbb{Z}$ and each nonzero $h\in\mathbb{Z}(p^\infty)$, there is an element $k\in\mathbb{Z}(p^\infty)$ such that $rk=h$. In an analogous fashion, an ideal I of a ring R is said to be *divided* if it is comparable with each principal ideal (see, for example, [3]).

Theorem 5.7. *Let R be a nonreduced ring with prime nilradical* $\operatorname{Nil}(R)$. *If* $\operatorname{Nil}(R)$ *is divided and $Z(R)$ properly contains* $\operatorname{Nil}(R)$, *then for each $r\in Z(R)\setminus\operatorname{Nil}(R)$, every power series with constant term r is a zero divisor of $R[\![\mathrm{x}]\!]$.*

Proof. Assume $\operatorname{Nil}(R)$ is divided and that $Z(R)$ properly contains $\operatorname{Nil}(R)$. Then, since $\operatorname{Nil}(R)$ is prime, $(0:r)\subset\operatorname{Nil}(R)$ for each $r\in Z(R)\setminus\operatorname{Nil}(R)$. For such an r, take a power series $t(\mathrm{x})=\sum r_i\mathrm{x}^i$ with constant term $r_0=r$. To build a nonzero annihilator of $t(\mathrm{x})$, start with a nonzero nilpotent $n_0\in(0:r)$. Since r is not nilpotent and $\operatorname{Nil}(R)$ is a divided prime, there is a nilpotent n_1 such that $rn_1=-r_1n_0$. Recursively find nilpotents n_j such that $rn_j=-(r_1n_{j-1}+r_2n_{j-2}+\cdots+r_jn_0)$ and set $n(\mathrm{x})=\sum n_i\mathrm{x}^i$. The product $t(\mathrm{x})n(\mathrm{x})=0$ since $r_0n_j+r_1n_{j-1}+\cdots+r_jn_0=0$ for each j. Since $n_0\neq 0$, $n(\mathrm{x})$ is a nonzero annihilator of $t(\mathrm{x})$. □

Theorem 5.8. *Let R be a nonreduced ring with prime nilradical* $\operatorname{Nil}(R)$. *If* $\operatorname{Nil}(R)$ *is divided and $Z(R)$ properly contains* $\operatorname{Nil}(R)$, *then* $\operatorname{diam}(\Gamma(R[\![\mathrm{x}]\!]))=3$.

Proof. If $Z(R)$ properly contains $\operatorname{Nil}(R)$, then there is a nonnilpotent zero divisor r and a nonzero nilpotent n such that $rn=0$. By the previous theorem, $t(\mathrm{x})=r+\mathrm{x}$ is a zero divisor of $R[\![\mathrm{x}]\!]$. Obviously, $\mathrm{x}=(r+\mathrm{x})-r$ is not a zero divisor. Thus $\operatorname{diam}(\Gamma(R[\![\mathrm{x}]\!]))=3$. □

Given any integral domain D that is not a field, we can use idealization to create a pair of rings R and S which have the same behavior as $\mathbb{Z}(+)\mathbb{Z}_p$ and $\mathbb{Z}(+)\mathbb{Z}(p^\infty)$ with regard to diameters.

Example 5.9. Let M be a maximal ideal of an integral domain D that is not a field and let J the injective hull of the D_M-module D/M. Then $R=D(+)D/M$ is a nonreduced ring with $\operatorname{diam}(\Gamma(R))=\operatorname{diam}(\Gamma(R[\mathrm{x}]))=\operatorname{diam}(\Gamma(R[\![\mathrm{x}]\!]))=2$, and $S=D(+)J$ is a nonreduced ring with $\operatorname{diam}(\Gamma(S))=\operatorname{diam}(\Gamma(S[\mathrm{x}]))=2$ and $\operatorname{diam}(\Gamma(S[\![\mathrm{x}]\!]))=3$. As in Example 5.6, $Z(R)\operatorname{Nil}(R)=(0)$ and $Z(S)\operatorname{Nil}(S)\neq(0)$.

Proof. As in Example 5.6, $Z(R)=M(+)D/M$ and $Z(S)=M(+)J$. Since D/M embeds in J and $M(D/M)=(0)$, both $Z(R)$ and $Z(S)$ have nonzero annihilators. On the other hand, if a,b are distinct nonzero elements of M, then $(a,0)(b,0)=(ab,0)\neq(0,0)$. Hence $d((a,0),(b,0))=2$. Both R and S are McCoy rings, thus $\operatorname{diam}(\Gamma(R))=$

$\operatorname{diam}(\Gamma(R[\mathrm{x}])) = 2$ and $\operatorname{diam}(\Gamma(S)) = \operatorname{diam}(\Gamma(S[\mathrm{x}])) = 2$. As an injective module is divisible, $(0)(+)J$ is a divided prime of S. Thus $\operatorname{diam}(\Gamma(S[\![\mathrm{x}]\!])) = 3$ by Theorem 5.8.

Let $r(\mathrm{x}) = \sum r_i \mathrm{x}^i$ be a zero divisor of $R[\![\mathrm{x}]\!]$ with each $r_i = (s_i, b_i)$. Without loss of generality we may assume $r_0 \neq 0$. If $r_0, r_1, \ldots, r_n$ are all nilpotent, while r_{n+1} is not, then $r(\mathrm{x}) - (r_0 + r_1\mathrm{x} + \cdots + r_n\mathrm{x}^n)$ is a zero divisor whose first nonzero term is not. Thus we may also assume that r_0 is not nilpotent. Hence it has the form (s_0, b_0) where s_0 is a nonzero element of M. Let $t(\mathrm{x}) = \sum t_i \mathrm{x}^i$ be a nonzero annihilator of $r(\mathrm{x})$. Again we may assume t_0 is not zero and we may write $t_i = (u_i, c_i)$. Since D is an integral domain, u_0 must be zero. Thus having $0 = r_0t_1 + r_1t_0 = (s_0, b_0)(u_1, c_1) + (s_1, b_1)(0, c_0) = (s_0u_1, s_1c_0)$, forces us to have both $u_1 = 0$ and $s_1 \in M$. Inductively, we get $u_k = 0$ and $s_k \in M$. Hence each coefficient of $r(\mathrm{x})$ is in $M(+)D/M = Z(R)$. Therefore, $\operatorname{diam}(\Gamma(R[\![\mathrm{x}]\!])) = 2$. □

Our last result provides a condition which is sufficient to give $\operatorname{diam}(\Gamma(R[\![\mathrm{x}]\!])) = 2$ when R is nonreduced with nonnilpotent zero divisors.

Theorem 5.10. *Let R be a nonreduced ring such that $Z(R)$ is not the nilradical of R. If $Z(R)$ has a nonzero annihilator, then $Z(R)$ is an ideal of R, R is a McCoy ring,* $\operatorname{diam}(\Gamma(R)) = \operatorname{diam}(\Gamma(R[\mathrm{x}])) = 2$ *and* $Z(R)[\![\mathrm{x}]\!] \subseteq Z(R[\![\mathrm{x}]\!])$. *Moreover, if* $Z(R)\operatorname{Nil}(R) = (0)$, *then* $\operatorname{diam}(\Gamma(R[\![\mathrm{x}]\!])) = 2$.

Proof. Assume $Z(R)$ has a nonzero annihilator. Since $Z(R)$ is not the nilradical, such an annihilator must be a nonzero nilpotent. Let n be such a nilpotent. Clearly, $Z(R)$ must be the annihilator of n. Thus $Z(R)$ is an ideal of R and R is a McCoy ring. Thus by Theorem 3.4, $\operatorname{diam}(\Gamma(R)) = \operatorname{diam}(\Gamma(R[\mathrm{x}])) = 2$.

In addition to the assumptions above, assume that $Z(R)\operatorname{Nil}(R) = (0)$. Clearly if $f(\mathrm{x}), g(\mathrm{x}) \in Z(R)[\![\mathrm{x}]\!]$, then they have a common nonzero annihilator—simply choose any nonzero nilpotent of R. Thus $d(f(\mathrm{x}), g(\mathrm{x})) \leqslant 2$. Hence all we need to do to complete the proof is to show that each zero divisor of $R[\![\mathrm{x}]\!]$ is contained in $Z(R)[\![\mathrm{x}]\!]$. Let $r(\mathrm{x})$ be a nonzero zero divisor of $R[\![\mathrm{x}]\!]$ and let $s(\mathrm{x})$ be a nonzero power series such that $r(\mathrm{x})s(\mathrm{x}) = 0$. Without loss of generality, we may assume both r_0 and s_0 are nonzero, but each must be a zero divisor of R. By [6, Proposition 3.5], the product $c(r)c(s)$ is contained in $\operatorname{Nil}(R)$. By way of contradiction, assume some r_i is not a zero divisor of R. Then each coefficient of $s(\mathrm{x})$ is nilpotent. Moreover, if i is the smallest positive integer for which r_i is not a zero divisor, then $r_js_k = 0$ for each $j < i$ and each k since $Z(R)\operatorname{Nil}(R) = (0)$. In particular, the coefficient on x^i in the product $r(\mathrm{x})s(\mathrm{x})$ is simply r_is_0. As $r(\mathrm{x})s(\mathrm{x}) = 0$, this is impossible since s_0 is not zero and r_i is not a zero divisor. Hence $r(\mathrm{x}) \in Z(R)[\![\mathrm{x}]\!]$ and therefore $\operatorname{diam}(\Gamma(R[\![\mathrm{x}]\!])) = 2$. □

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